A Note On Computing the Drazin Inverse

Nicholas J. Rose
Department of Mathematics
North Carolina State University
Raleigh, North Carolina 27607

Submitted by Robert J. Plemmons

ABSTRACT

A method is given for computing the Drazin inverse of a square matrix A of order n as a polynomial in A of degree $n-1$ or less from the characteristic polynomial of A.

The Drazin (pseudo) inverse of a square matrix has recently been used in the solution of systems of differential or difference equations with singular constant coefficient matrices [1]. Methods for computing the Drazin inverse are therefore of some interest. In [1] a method is given for computing the Drazin inverse of an $n\times n$ matrix A as a polynomial in A of degree $n-1$ or less provided the eigenvalues of A are known. In this note we show how to find the Drazin inverse using only the characteristic (or minimal) polynomial. Greville [2] has given an expression for the Drazin inverse of A as a polynomial in A. However, the Greville expression is usually of degree higher than $n-1$ and must be reduced using the Cayley-Hamilton theorem. Our approach always yields a polynomial of degree $n-1$ or less.

Let A be an arbitrary square matrix of order n with complex elements and characteristic polynomial given by

$$c(\lambda)=\lambda^{l}(\lambda^{s}-p_1\lambda^{s-1}-\cdots-p_s), \tag{1}$$

where $p_s\neq 0$ and $s+l=n$. A similarity transformation can be applied to A to yield

$$A=T\begin{bmatrix} C & 0\\ 0 & N\end{bmatrix}T^{-1}, \tag{2}$$

where N is a nilpotent matrix of order l and C is a nonsingular matrix of order s. The Drazin inverse is given by [3]

$$A^d = T\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} T^{-1}. \tag{3}$$

Since C is nonsingular, there exists a polynomial p of degree $s-1$ or less such that

$$C^{-l-1} = p(C). \tag{4}$$

It is easily verified from (1) and (2) that

$$A^d = A^l p(A), \tag{5}$$

which is a polynomial of degree $n-1$ or less.

If the index of A ($=$index of N) is known and equals $k \leqslant l$, we need only calculate

$$C^{-k-1} = q(C), \tag{6}$$

where the degree of q is $s-1$ or less, and we can easily verify that

$$A^d = A^k q(A). \tag{7}$$

In order to compute the Drazin inverse from the characteristic equation it is only necessary to compute the polynomial p in (4). Since the characteristic equation for the matrix C in (1) is

$$\lambda^s - p_1\lambda^{s-1} - \cdots - p_s = 0, \qquad p_s \neq 0, \tag{8}$$

the polynomial p is just the polynomial having the property that

$$\lambda^{-l-1} = p(\lambda) \tag{9}$$

holds for every root of (8). Thus to find p we may use Eq. (8) to calculate successively $\lambda^{-1}, \lambda^{-2}, \dots, \lambda^{-l-1} = p(\lambda)$. [If the index of A is known and equals $k \leqslant l$, then it is only necessary to compute $\lambda^{-k-1} = q(\lambda)$]. This procedure is illustrated in Example 1. The next two examples illustrate how the algebra can be shortened in certain simple cases.

EXAMPLE 1. $c(\lambda)=\lambda^2(\lambda^2+5\lambda+1)$.
The characteristic equation of C is

$$\lambda^2+5\lambda+1=0.$$

Therefore

$$\lambda^{-1}=-\lambda-5,$$

$$\lambda^{-2}=-1-5\lambda^{-1}=5\lambda+24,$$

$$p(\lambda)=\lambda^{-3}=5+24\lambda^{-1}=-24\lambda-115,$$

$$A^d=A^2p(A)=-A^2(24A+115I).$$

EXAMPLE 2. $c(\lambda)=\lambda^2(\lambda^4+\lambda^3+\lambda^2+\lambda+1)$.
Since $\lambda\neq 1$,

$$\lambda^4+\lambda^3+\lambda^2+\lambda+1=\frac{\lambda^5-1}{\lambda-1}.$$

Therefore

$$\lambda^5=1,$$

$$\lambda^{-5}=1,$$

$$p(\lambda)=\lambda^{-3}=\lambda^2,$$

$$A^d=A^2p(A)=A^4.$$

EXAMPLE 3. $c(\lambda)=\lambda^4(\lambda-1)^3$.

$$p(\lambda)=\lambda^{-5}=\left[1+(\lambda-1)\right]^{-5}, \qquad \text{where} \quad (\lambda-1)^3=0.$$

Using the binomial theorem, we have

$$p(\lambda)=1-5(\lambda-1)+15(\lambda-1)^2,$$

$$A^d=A^4p(A)=A^4\left[I-5(A-I)+15(A-I)^2\right].$$

It is of course not a simple matter to compute the characteristic polynomial or the index of a matrix of high order. One method of determining the characteristic polynomial is the Souriau-Frame algorithm described by Greville [2]. Although not explicitly pointed out by Greville, it is implicit in his paper that the Souriau-Frame algorithm also produces the index of the matrix. We briefly indicate how this can be done.

Let A be a square matrix of order n. In the Souriau-Frame algorithm, a sequence of numbers $p_1,\dots,p_n$ and a sequence of matrices $B_0,\dots,B_n$ are defined by

$$B_0 = I, \qquad p_j = j^{-1}\operatorname{tr}(AB_{j-1}), \qquad B_j = AB_{j-1} - p_j I$$

for $j = 1,2,\dots,n$. Then the characteristic polynomial of A is given by

$$c(\lambda) = \lambda^n - p_1\lambda^{n-1} - \cdots - p_n.$$

If r is the smallest integer such that $B_r = 0$ and s is the largest integer such that $p_s \neq 0$, then it is easy to show that an annihilating polynomial for A is

$$B_r = A^{r-s}B_s,$$

where $r-s$ is the index of A.

REFERENCES

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